Distinguished principal series representations for GLn over a p-adic field

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1 Introduction

For K/F a quadratic extension of p-adic fields, let σ be the conjugation relative to this extension, and $\eta_{K/F}$ be the character of F^* with kernel norms of K^*.
If π is a smooth irreducible representation of $GL(n,K)$, and χ a character of F^*, the dimension of the space of linear forms on its space, which transform by χ under $GL(n,F)$ (with respect to the action $[(L,g)\mapsto L\circ\pi(g)]$) , is known to be at most one (Proposition 11, [F]). One says that π is χ-distinguished if this dimension is one, and says that π is distinguished if it is 1-distinguished.
In this article, we give a description of distinguished principal series representations of $GL(n,K)$. The result (Theorem 3.2) is that the the irreducible distinguished representations of the principal series of $GL(n,K)$ are (up to isomorphism) those unitarily induced from a character $\chi=(\chi_1,...,\chi_n)$ of the maximal torus of diagonal matrices, such that there exists $r\leq n/2$, for which $\chi_{i+1}^{\sigma}={\chi_i}^{-1}$ for $i=1,3,..,2r-1$, and $\chi_{i|F^*}=1$ for $i>2r$. For the quadratic extension $\mathbb{C}/\mathbb{R}$, it is known (cf.[P]) that the analogous result is true for tempered representations.
For $n\geq 3$, this gives a counter-example (Corollary 3.1) to a conjecture of Jacquet (Conjecture 1 in [A]). This conjecture states that an irreducible representation π of $GL(n,K)$ with central character trivial on F^* is isomorphic to $\check{\pi}^{\sigma}$ if and only if it is distinguished or $\eta_{K/F}$-distinguished (where $\eta_{K/F}$ is the character of order 2 of F^*, attached by local class field theory to the extension K/F). For discrete series representations, the conjecture is verified, it was proved in [K].
Unitary irreducible distinguished principal series representations of $GL(2,K)$ were described in [H], and the general case of distinguished irreducible principal series representations of $GL(2,K)$ was treated in [F-H]. We use this occasion to give a different proof of the result for $GL(2,K)$ than the one in [F-H]. To do this, in Theorems 4.1 and 4.2, we extend a criterion of Hakim (th.4.1, [H]) characterising smooth unitary irreducible distinguished representations of $GL(2,K)$ in terms of γ factors at 1/2, to all smooth irreducible distinguished representations of $GL(2,K)$.

2 Preliminaries

Let ϕ be a group automorphism, and x an element of the group, we sometimes note x^{ϕ} instead of $\phi(x)$, and $x^{-\phi}$ the inverse of x^{ϕ}. If $\phi=x\mapsto h^{-1}xh$ for h in the group, then x^h designs x^{ϕ}.

Let G be a locally compact totally disconnected group, H a closed subgroup of G.
We note Δ_G the module of G, given by the relation $d_G(gx) = \Delta_G(g)d_G(x)$, for a right Haar measure d_G on G.
Let X be a locally closed subspace of G, with $H.X \subset X$. If V is a complex vector space, we note $D(X, V)$ the space of smooth V-valued functions on X with compact support (if $V = \mathbb{C}$, we simply note it $D(X)$).
Let ρ be a a smooth representation of H in a complex vector space V_ρ, we note $D(H\backslash X, \rho, V_\rho)$ the space of smooth V_ρ-valued functions f on X, with compact support modulo H, which verify $f(hx) = \rho(h)f(x)$ for $h \in H$ and $x \in X$ (if ρ is a character, we note it $D(H\backslash X, \rho)$).
We note $ind_H^G(\rho)$ the representation by right translations of G in
$D(H\backslash G, (\Delta_G/\Delta_H)^{1/2}\rho, V_\rho)$.

Let F be a non archimedean local field of characteristic zero, and K a quadratic extension of F. We have $K = F(\delta)$ with δ^2 in F^*.
We note $| \ |_K$ and $| \ |_F$ the modules of K and F respectively.
We note σ the non trivial element of the Galois group $G(K/F)$ of K over F, and we use the same letter to design for its action on $M_n(K)$.
We note $N_{K/F}$ the norm of the extension K/F and we note $\eta_{K/F}$ the nontrivial character of F^* which is trivial on $N_{K/F}(K^*)$.
Whenever G is an algebraic group defined over F, we note $G(K)$ its K-points and $G(F)$ its F-points.

The group $GL(n)$ will be noted G_n, its standard Borel subgroup will be noted B_n, its unipotent radical U_n, and the standard maximal split torus of diagonal matrices T_n.

We note S the space of matrices M in $G_n(K)$ satisfying $MM^\sigma = 1$.

Everything in this paragraph is more or less contained in [F1], we give detailed proofs here for convenience of the reader.

Proposition 2.1. *([S], ch.10, prop.3)*
We have a homeomorphism between $G_n(K)/G_n(F)$ and S given by the map $S_n : g \mapsto g^\sigma g^{-1}$.

Proposition 2.2. *For its natural action on S, each orbit of $B_n(K)$ contains one and only one element of $\mathfrak{S}_n$ of order 2 or 1.*

Proof. We begin with the following:

Lemma 2.1. *Let w be an element of $\mathfrak{S}_n \subset G_n(K)$ of order at most 2.*
Let θ' be the involution of $T_n(K)$ given by $t \mapsto w^{-1}t^\sigma w$, then any $t \in T_n(K)$ with $t\theta'(t) = 1$ is of the form $a/\theta'(a)$ for some $a \in T_n(K)$.

Proof of Lemma 2.1. There exists $r \leq n/2$ such that up to conjugacy, w is
$(1, 2)(3, 4)...(2r - 1, 2r)$.

We write $t = \begin{pmatrix} z_1 & & & & & & & \\ & {z_1}' & & & & & & \\ & & \ddots & & & & & \\ & & & z_r & & & & \\ & & & & {z_r}' & & & \\ & & & & & z_{2r+1} & & \\ & & & & & & \ddots & \\ & & & & & & & z_n \end{pmatrix}$, hence for $i \leq r$, we have $z_i\sigma(z_i') = 1$,
and $z_j\sigma(z_j) = 1$ for $j \geq 2r+1$.
Hilbert's Theorem 90 asserts that each $z_j, j \geq 2r+1$ is of the form $u_{j-2r}/\sigma(u_{j-2r})$, for some $u_{j-2r} \in K^*$.

We then take $a = \begin{pmatrix} z_1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & z_r & & & & \\ & & & & 1 & & & \\ & & & & & u_1 & & \\ & & & & & & \ddots & \\ & & & & & & & u_{n-2r} \end{pmatrix}$. □

Lemma 2.2. *Let N be an algebraic connected unipotent group over K. Let θ be an involutive automorphism of $N(K)$. If $x \in N(K)$, verifies $x\theta(x) = 1_N$ then, there is $a \in N$ such that $x = \theta(a^{-1})a$.*

Proof of Lemma 2.2: The group $N(K)$ has a composition series $1_N = N_0 \subset N_1 \subset ...N_{n-1} \subset N_n = N(K)$, such that each quotient N_{i+1}/N_i is isomorphic to $(K,+)$, and each commutator subgroup $[N, N_{i+1}]$ is a subgroup of N_i.
Now we prove the Lemma by induction on n:
If $n = 1$, then $N(K)$ is isomorphic to $(K,+)$, one concludes taking $a = x/2$.
$n \mapsto n+1$:
suppose the Lemma is true for every $N(K)$ of length n.
Let $N(K)$ be of length $n+1$, we note $\bar{x}$ the class of x in $N(K)/N_1$.
By induction hypothesis, one gets that there exists an element in $h \in N_1$, and an element u in $N(K)$ such that $x = \theta(u^{-1})uh$.
Here h lies in the center of $N(K)$, because $[N(K), N_1] = 1_N$.
As $x\theta(x) = 1$, we get $h\theta(h) = 1$. By induction hypothesis again, we get $h = \theta(b^{-1})b$ for $b \in N_1$.
We then take $a = ub$. □

We get back to the proof of the Proposition 2.2.

For w in $\mathfrak{S}_n$, one notes U_w the subgroup of U_n generated by the elementary subgroups U_α, with α positive, and $w\alpha$ negative, and ${U_w}'$ the subgroup of U_n generated by the elementary subgroups U_α, with α positive, and $w\alpha$ positive. Then $U_n = {U_w}'U_w$.
Let s be in S. According to Bruhat's decomposition, there is w in $\mathfrak{S}_n$, and a in $T_n(K)$, n_1 in $U_n(K)$ and ${n_2}^+$ in U_w, such that $s = n_1 a w {n_2}^+$, with unicity of the decomposition.

Then $s = s^{-\sigma} = {n_2^+}^{-\sigma} w^{-1} a^{-\sigma} {n_1}^{-\sigma}$.
Thus we have $aw = (aw)^{-\sigma}$, i.e. $w^2 = 1$ and $a^w = a^{-\sigma}$.
Now we write ${n_1}^{-\sigma} = u^- u^+$ with $u^- \in {U_w}'$ and $u^+ \in U_w$, comparing s and $s^{-\sigma}$, u^+ must be equal to ${n_2}^+$.
Hence $s = n_1 a w {u^-}^{-1} {n_1}^{-\sigma}$, thus we suppose $s = awn$, with n in U'_w.
From $s = s^{-\sigma}$, one has the relation $awn(aw)^{-1} = n^{-\sigma}$, applying σ on each side, this becomes $(aw)^{-1} n^{\sigma} aw = n^{-1}$.
But $\theta : u \mapsto (aw)^{-1} u^{\sigma} aw$ is an involutive automorphism of U'_w, hence from Lemma 2.2, there is u' in U'_w such that $n = \theta(u^{-1})u$.
This gives $s = u^{-\sigma} awu$, so that we suppose $s = aw$. Again $wa^{\sigma}w = a^{-1}$, and applying Lemma 2.1 to $\theta' : x \mapsto wx^{\sigma}w$, we deduce that a is of the form $y\theta'(y^{-1})$, and $s = ywy^{-\sigma}$. □

Let u be the element $\begin{pmatrix} 1 & -\delta \\ 1 & \delta \end{pmatrix}$ of $M_2(K)$; one has $S_2(u) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (cf. Proposition 2.1).
We notice for further use (cf. proof of Proposition 3.1), that if we note $\tilde{T}$ the subgroup $\left\{ \begin{pmatrix} z & 0 \\ 0 & z^{\sigma} \end{pmatrix} \in G_2(K) | z \in K^* \right\}$, then $u^{-1}\tilde{T}u = T = \left\{ \begin{pmatrix} x & \Delta y \\ y & x \end{pmatrix} \in G_2(F) | x, y \in F \right\}$.

For $r \leq n/2$, one notes U_r the $n \times n$ matrix given by the following block decomposition:
$$\begin{pmatrix} u & & & \\ & \ddots & & \\ & & u & \\ & & & I_{n-2r} \end{pmatrix}$$
If w is an element of $\mathfrak{S}_n$ naturally injected in $G_n(K)$, one notes $U_r^w = w^{-1} U_r w$.

Corollary 2.1. *The elements U_r^w for $0 \leq r \leq n/2$, and $w \in \mathfrak{S}_n$ give a complete set of representatives of classes of $B_n(K) \backslash G_n(K) / G_n(F)$.*

Let $G_n = \coprod_{w \in \mathfrak{S}_n} B_n w B_n$ be the Bruhat decomposition of G_n. We call a double-class BwB a Bruhat cell.

Lemma 2.3. *One can order the Bruhat cells C_1, C_2,..., $C_{n!}$ so that for every $1 \leq i \leq n!$, the cell C_i is closed in $G_n - \coprod_{k=1}^{i-1} C_i$.*

Proof. Choose $C_1 = B_n$. It is closed in G_n. Now let w_2 be an element of $\mathfrak{S}_n - Id$, with minimal length. Then from 8.5.5. of [Sp], one has that the Bruhat cell Bw_2B is closed in $G_n - B_n$ with respect to the Zariski topology, hence for the p-adic topology, we call it C_2. We conclude by repeating this process. □

Corollary 2.2. *One can order the classes A_1,..., A_t of $B_n(K) \backslash G_n(K) / G_n(F)$, so that A_i is closed in $G_n(K) - \coprod_{k=1}^{i-1} A_i$.*

Proof. From the proof of Proposition 2.2, we know that if C is a Bruhat cell of G_n, then $S_n \cap C$ is either empty, or it corresponds through the homemorphism S_n to a class A of $B_n(K) \backslash G_n(K) / G_n(F)$. The conclusion follows the preceeding Lemma. □

Corollary 2.3. *Each A_i is locally closed in $G_n(K)$ for the Zariski topology.*

We will also need the following Lemma:

Lemma 2.4. *Let G, H,X, and (ρ, V_ρ) be as in the beginning of the section, the map Φ from $D(X)\otimes V_\rho$ to $D(H\backslash X,\rho,V_\rho)$ defined by $\Phi : f\otimes v \mapsto (x \mapsto \int_H f(hx)\rho(h^{-1})vdh)$ is surjective.*

Proof. Let $v \in V_\rho$, U an open subset of G that intersects X, small enough for $h \mapsto \rho(h)v$ to be trivial on $H \cap UU^{-1}$.
Let f' be the function with support in $H(X \cap U)$ defined by $hx \mapsto \rho(h)v$.
Such functions generate $D(H\backslash X, \rho, V_\rho)$ as a vector space.
Now let f be the function of $D(X, V_\rho)$ defined by $x \mapsto 1_{U\cap X}(x)v$, then $\Phi(f)$ is a multiple of f'.
But for x in $U \cap X$, $\Phi(f)(x) = \int_H \rho(h^{-1})f(hx)dh$ because $h \mapsto \rho(h)v$ is trivial on $H \cap UU^{-1}$, plus $h \mapsto f(hx)$ is a positive function that multiplies v, and $f(x) = V$, so $F(f)(x)$ is v multiplied by a strictly positive scalar. $\square$

Corollary 2.4. *Let Y be a closed subset of X, H-stable, then the restriction map from $D(H\backslash X, \rho, V_\rho)$ to $D(H\backslash Y, \rho, V_\rho)$ is surjective.*

Proof. This is a consequence of the known surjectivity of the restriction map from $D(X)$ to $D(Y)$, which implies the surjectivity of the restriction from $D(X, V_\rho)$ to $D(Y, V_\rho)$ and of the commutativity of the diagram:

$$\begin{array}{ccc} D(X) & \rightarrow & D(Y) \\ \downarrow\Phi & & \downarrow\Phi \\ D(H\backslash X,\rho) & \rightarrow & D(H\backslash Y,\rho) \end{array}.$$

$\square$

3 Distinguished principal series

If π is a smooth representation of $G_n(K)$ of space V_π, and χ is a character of F^*, we say that π is χ-distinguished if there exists on V_π a nonzero linear form L such that $L(\pi(g)v) = \chi(det(g))L(v)$ whenever g is in $G_n(F)$ and v belongs to V_π. If χ is trivial, we simply say that π is distinguished.

We first recall the following:

Theorem 3.1. *([F], Proposition 12)*
Let π be a smooth irreducible distinguished representation of $G_n(K)$, then $\pi^\sigma \simeq \check{\pi}$.

Let $\chi_1, ..., \chi_n$ be n characters of K^*, with none of their quotients equal to $|\ |_K$. We note χ the character of $B_n(K)$ defined by $\chi\begin{pmatrix} b_1 & \star & \star \\ & \ddots & \star \\ & & b_n \end{pmatrix} = \chi_1(b_1)...\chi_n(b_n)$.
We note $\pi(\chi)$ the representation of $G_n(K)$ by right translation on the space of functions $D(B_n(K)\backslash G_n(K), {\Delta_{B_n}}^{-1/2}\chi)$.
This representation is smooth, irreducible and called the principal series attached to χ.
If π is a smooth representation of $G_n(K)$, we note $\check{pi}$ its smooth contragredient.

We will need the following Lemma:

Lemma 3.1. *(Proposition 26 in [F1]) Let $\bar{m} = (m_1, \ldots, m_l)$ be a partition of a positive integer m, let $P_{\bar{m}}$ be the corresponding standard parabolic subgroup, and for each $1 \le i \le l$, let π_i be a smooth* *distinguished representation of* $G_{m_i}(K)$*, then* $\pi_1 \times \cdots \times \pi_l = ind_{P_{\bar{m}}(K)}^{G_m(K)}(\Delta_{P_{\bar{m}}(K)}^{-1/2}(\pi_1 \otimes \cdots \otimes \pi_l))$ *is distinguished.*

We now come to the principal results:

Proposition 3.1. *Let* $\chi = (\chi_1, \ldots, \chi_n)$ *be a character of* $T_n(K)$*, suppose that the principal series representation* $\pi(\chi)$ *is distinguished, there exists a re-ordering of the* χ_i*'s, and* $r \leq n/2$*, such that* $\chi_{i+1}^{\sigma} = {\chi_i}^{-1}$ *for* $i = 1, 3, .., 2r-1$*, and that* $\chi_{i|F^*} = 1$ *for* $i > 2r$*.*

Proof. We write $B = B_n(K)$, $G = G_n(K)$. We have from Corollary 2.2 and 2.4 the following exact sequence of smooth $G_n(F)$-modules:

$$D(B\backslash G - A_1, \Delta_B^{-1/2}\chi) \hookrightarrow D(B\backslash G, \Delta_B^{-1/2}\chi) \rightarrow D(B\backslash A_1, \Delta_B^{-1/2}\chi).$$

Hence there is a non zero distinguished linear form either on $D(B\backslash A_1, \Delta_B^{-1/2}\chi)$, or on $D(B\backslash G - A_1, \Delta_B^{-1/2}\chi)$.
In the second case we have the exact sequence

$$D(B\backslash G - A_1 \sqcup A_2, \Delta_B^{-1/2}\chi) \hookrightarrow D(B\backslash G - A_1, \Delta_B^{-1/2}\chi) \rightarrow D(B\backslash A_2, \Delta_B^{-1/2}\chi).$$

Repeating the process, we deduce the existence of a non zero distinguished linar form on one of the spaces $D(B\backslash A_i, \Delta_B^{-1/2}\chi)$.
From Corollary 2.1, we choose w in S_n and $r \leq n/2$ such that $A_i = BU_r^w G_n(F)$. The application $f \mapsto [x \mapsto f(U_r^w x)]$ gives an isomorphism of $G_n(F)$-modules between $D(B\backslash A_i, \Delta_B^{-1/2}\chi)$ and $D(U_r^{-w}BU_r^w \cap G_n(F)\backslash G_n(F), \Delta'\chi')$ where $\Delta'(x) = \Delta_B^{-1/2}(U_r^w x U_r^{-w})$ and $\chi'(x) = \chi(U_r^w x U_r^{-w})$.
Now there exists a nonzero $G_n(F)$-invariant linear form on $D(U_r^{-w}BU_r^w \cap G_n(F)\backslash G_n(F), \Delta'\chi')$ if and only if $\Delta'\chi'$ is equal to the inverse of the module of $U_r^{-w}BU_r^w \cap G_n(F)$ (cf.[B-H], ch.1, prop.3.4). From this we deduce that χ' is positive on $U_r^{-w}BU_r^w \cap G_n(F)$ or equivalently χ is positive on $B \cap U_r^w G_n(F) U_r^{-w}$.
Let $\bar{T}_r$ be the F-torus of matrices of the form

$$\begin{pmatrix} z_1 & & & & & & & \\ & {z_1}^{\sigma} & & & & & & \\ & & \ddots & & & & & \\ & & & z_r & & & & \\ & & & & {z_r}^{\sigma} & & & \\ & & & & & x_1 & & \\ & & & & & & \ddots & \\ & & & & & & & x_t \end{pmatrix}$$

with $2r + t = n$, $z_i \in K^*$, $x_i \in F^*$, then one has ${\bar{T}_r}^w \subset B \cap U_r^w G_n(F) U_r^{-w}$, so that χ must be positive on ${\bar{T}_r}^w$.
We remark that if χ is unitary, then χ is trivial on ${\bar{T}_r}^w$, and $\pi(\chi)$ is of the desired form.
For the general case, we deduce from Theorem 3.1, that there exists three integers $p \geq 0, q \geq 0, s \geq 0$ such that up to reordering, we have $\chi_{2i} = \chi_{2i-1}^{-\sigma}$ for $1 \leq i \leq p$, we have $\chi_{2p+k|F^*} = 1$ for $1 \leq k \leq q$

and these χ_{2p+k}'s are different (so that $\chi_{2p+k} \neq \chi_{2p+k'}^{-\sigma}$ for $k \neq k'$), and $\chi_{2p+q+j|F^*} = \eta_{K/F}$ for $1 \leq j \leq s$, these χ_{2p+q+j}'s being different.
We note $\mu_k = \chi_{2p+k}$ for $q \geq k \geq 1$, and $\nu'_k = \chi_{2p+q+k'}$ for $s \geq k' \geq 1$.
We show that if such a character χ is positive on a conjugate of $\bar{T}_r$ by an element of S_n, then $s = 0$. Supose ν_1 appears, then either ν_1 is positive on F^*, but that is not possible, or it is coupled with another χ_i, and (ν_1, χ_i) is positive on elements (z, z^σ), for z in K^*.
Suppose $\chi_i = \nu_j$ for some $j \neq 1$, then (ν_1, χ_i) is unitary, so it must be trivial on couples (z, z^σ), which implies $\nu_1 = {\nu_j}^{-\sigma} = \nu_j$, which is absurd.
The character χ_i cannot be of the form μ_j, because it would imply $\nu_{1|F^*} = 1$.
The last case is $i \leq 2p$, then ${\nu_1}^{-\sigma} = \nu_1$ must be the unitary part of χ_i because of the positivity of (ν_1, χ_i) on the couples (z, z^σ).
But ${\chi_i}^{-\sigma}$ also appears and is not trivial on F^*, hence must be coupled with another character χ_j with $j \leq 2p$ and $j \neq i$, such that $({\chi_i}^{-\sigma}, \chi_j)$ is positive on the elements (z, z^σ), for z in K^*, which implies that χ_j has unitary part ${\nu_1}^{-\sigma} = \nu_1$. The character χ_j cannot be a μ_k because of its unitary part.
If it is a χ_k with $k \leq 2p$, we consider again ${\chi_k}^{-\sigma}$.
But repeating the process lengthily enough, we can suppose that χ_j is of the form ν_k, for $k \neq 1$. Taking unitary parts, we see that $\nu_k = {\nu_1}^{-\sigma} = \nu_1$, which is in contradiction with the fact that all ν_i's are different. We conclude that $s = 0$. □

Theorem 3.2. *Let $\chi = (\chi_1, \ldots, \chi_n)$ be a character of $T_n(K)$, the principal series representation $\pi(\chi)$ is distinguished if and only if there exists $r \leq n/2$, such that $\chi_{i+1}^\sigma = {\chi_i}^{-1}$ for $i = 1, 3, .., 2r-1$, and that $\chi_{i|F^*} = 1$ for $i > 2r$.*

Proof. There is one implication left.
Suppose χ is of the desired form, then $\pi(\chi)$ is parabolically (unitarily) induced from representations of the type $\pi(\chi_i, {\chi_i}^{-\sigma})$ of $G_2(K)$, and distinguished characters of K^*.
Hence, because of Lemma 3.1 the Theorem will be proved if we know that the representations $\pi(\chi_i, {\chi_i}^{-\sigma})$ are distinguished, but this is Corollary 4.1 of the next paragraph. □

This gives a counter-example to a conjecture of Jacquet (conjecture 1 in [A]), asserting that if an irreducible admissible representation π of $G_n(K)$ verifies that $\check{\pi}$ is isomorphic to π^σ, then it is distinguished if n is odd, and it is distinguished or $\eta_{K/F}$-distinguished if n is even.

Corollary 3.1. *For $n \geq 3$, there exist smooth irreducible representations π of $G_n(K)$, with central character trivial on F^*, that are neither distinguished, nor $\eta_{K/F}$-distinguished, but verify that $\check{\pi}$ is isomorphic to π^σ.*

Proof. Take $\chi_1, \ldots, \chi_n$, all different, such that $\chi_{1|F^*} = \chi_{2|F^*} = \eta_{K/F}$, and $\chi_{j|F^*} = 1$ for $3 \leq j \leq n$. Because each χ_i has trivial restriction to $N_{K/F}(K^*)$, it is equal to $\chi_i^{-\sigma}$, hence $\check{\pi}$ is isomorphic to π^σ. Another consequence is that if k and l are two different integers between 1 and n, then $\chi_k \neq \chi_l^{-\sigma}$, because we supposed the χ_i's all different.
Then it follows from Theorem 3.2 that $\pi = \pi(\chi_1, \ldots, \chi_n)$ is neither distinguished, nor $\eta_{K/F}$-distinguished, but clearly, the central character of π is trivial on F^* and $\check{\pi}$ is isomorphic to π^σ. □

4 Distinction and gamma factors for $GL(2)$

As said in the introduction, in this section we generalize to smooth infinite dimensional irreducible representations of $G_2(K)$ a criterion of Hakim (cf. [H], Theorem 4.1) characterising smooth unitary irreducible distinguished representations of $G_2(K)$. In proof of Theorem 4.1 of [H], Hakim deals with unitary representations so that the integrals of Kirillov functions on F^* with respect to a Haar measure of F^* converge. We skip the convergence problems using Proposition 2.9 of chapter 1 of [J-L].

We note $M(K)$ the mirabolic subgroup of $G_2(K)$ of matrices of the form $\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}$ with a in K^* and x in K, and $M(F)$ its intersection with $G_2(F)$. We note w the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
Let π be a smooth infinite dimensional irreducible representation of $G_2(K)$, it is known that it is generic (cf.[Z] for example). Let $K(\pi,\psi)$ be its Kirillov model corresponding to ψ ([J-L], th. 2.13), it contains the subspace $D(K^*)$ of functions with compact support on the group K^*. If ϕ belongs to $K(\pi,\psi)$, and x belongs to K, then $\phi - \pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\phi$ belongs to $D(K^*)$ ([J-L], prop.2.9, ch.1), from this follows that $K(\pi,\psi) = D(K^*) + \pi(w)D(K^*)$.

We now recall a consequence of the functional equation at 1/2 for Kirillov representations (cf. [B], section 4.7).
Forall ϕ in $K(\pi,\psi)$ and χ character of K^*, we have whenever both sides converge absolutely:

$$\textstyle\int_{K^*} \pi(w)\phi(x)(c_\pi\chi)^{-1}(x)d^*x = \gamma(\pi\otimes\chi,\psi)\int_{K^*}\phi(x)\chi(x)d^*x \tag{1}$$

where d^*x is a Haar measure on K^*, and c_π is the central character of π.

Theorem 4.1. *Let π be a smooth irreducible representation of $G_2(K)$ of infinite dimension with central character trivial on F^*, and ψ a nontrivial character of K trivial on F. If $\gamma(\pi\otimes\chi,\psi) = 1$ for every character χ of K^* trivial on F^*, then π is distinguished.*

Proof. In fact, using a Fourier inversion in functional equation 1 and the change of variable $x \mapsto x^{-1}$, we deduce that forall ϕ in $D(K^*)\cap\pi(w)D(K^*)$, we have

$$c_\pi(x)\int_{F^*}\pi(w)\phi(tx^{-1})d^*t = \int_{F^*}\phi(tx)d^*t$$

(d^*t is a Haar measure on F^*) which for $x=1$ gives

$$\int_{F^*}\pi(w)\phi(t)d^*t = \int_{F^*}\phi(t)d^*t.$$

Now we define on $K(\pi,\psi)$ a linear form λ by:

$$\lambda(\phi_1 + \pi(w)\phi_2) = \int_{F^*}\phi_1(t)d^*t + \int_{F^*}\phi_2(t)d^*t$$

for ϕ_1 and ϕ_2 in $D(K^*)$, which is well defined because of the previous equality and the fact that $K(\pi,\psi) = D(K^*) + \pi(w)D(K^*)$.

It is clear that λ is w-invariant. As the central character of π is trivial on F^*, λ is also F^*-invariant. Because $GL_2(F)$ is generated by $M(F)$, its center, and w, it remains to show that λ is $M(F)$-invariant.
Since ψ is trivial on F, one has if $\phi \in D(K^*)$ and $m \in M(F)$ the equality $\lambda(\pi(m)\phi) = \lambda(\phi)$.
Now if $\phi = \pi(w)\phi_2 \in \pi(w)D(K^*)$, and if a belongs to F^*, then $\pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\pi(w)\phi_2 = \pi(w)\pi\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\phi_2 = \pi(w)\pi\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\phi_2$ because the central character of π is trivial on F^*, and $\lambda(\pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\phi) = \lambda(\phi)$.
If $x \in F$, then $\pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\phi - \phi$ is a function in $D(K^*)$, which vanishes on F^*, hence $\lambda\pi(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\phi - \phi) = 0$.

Eventually λ is $M(F)$-invariant, hence $G_2(F)$-invariant, it is clear that its restriction to $D(K^*)$ is non zero. $\square$

Corollary 4.1. *Let μ be a character of K^*, then $\pi(\mu, \mu^{-\sigma})$ is distinguished.*

Proof. indeed, first we notice that the central character $\mu\mu^{-\sigma}$ of $\pi(\mu,\mu^{-\sigma})$ is trivial on F^*.
Now let χ be a character of K^*/F^*, then $\gamma(\pi(\mu,\mu^{-\sigma})\otimes\chi,\psi) = \gamma(\mu\chi,\psi)\gamma(\mu^{-\sigma}\chi,\psi) = \gamma(\mu\chi,\psi)\gamma(\mu^{-1}\chi^{\sigma},\psi^{\sigma})$, and as $\psi|F = 1$ and $\chi_{|F^*} = 1$, one has $\psi^{\sigma} = \psi^{-1}$ and $\chi^{\sigma} = \chi^{-1}$, so that $\gamma(\pi(\chi,\chi^{-\sigma}),\psi) = \gamma(\mu\chi,\psi)\gamma(\mu^{-1}\chi^{-1},\psi^{-1}) = 1$. The conclusion falls from Proposition 4.1. $\square$

Assuming Theorem 1.2 of [A-G], the converse of Theorem 4.1 is also true:

Theorem 4.2. *Let π be a smooth irreducible representation of infinite dimension of $G_2(K)$ with central character trivial on F^* and ψ a non trivial character of K/F, it is distinguished if and only if $\gamma(\pi\otimes\chi,\psi) = 1$ for every character χ of K^* trivial on F^*.*

Proof. It suffices to show that if π is a smooth irreducible distinguished representation of infinite dimension of $G_2(K)$, and ψ a non trivial character of K/F, then $\gamma(\pi,\psi) = 1$. Suppose λ is a non zero $G_2(F)$-invariant linear form on $K(\pi,\psi)$, it is shown in the proof of the corollary of Proposition 3.3 in [H], that its restriction to $D(F^*)$ must be a multiple of the Haar measure on F^*. Hence for any function ϕ in $D(K^*) \cap \pi(w)D(K^*)$, we must have $\int_{F^*}\phi(t)d^*t = \int_{F^*}\pi(w)\phi(t)d^*t$.

From this one deduces that for any function in $D(K^*) \cap \pi(w)D(K^*)$:

$$\begin{aligned}
\int_{K^*} \pi(w)\phi(x)c_\pi^{-1}(x)d^*x &= \int_{K^*/F^*} c_\pi^{-1}(a)\int_{F^*} \pi(w)\phi(ta)d^*tda \\
&= \int_{K^*/F^*} c_\pi^{-1}(a)\int_{F^*} \pi(\begin{matrix} a & 0 \\ 0 & 1 \end{matrix})\pi(w)\phi(t)d^*tda \\
&= \int_{K^*/F^*} c_\pi^{-1}(a)\int_{F^*} \pi(w)c_\pi(a)\pi(\begin{matrix} a^{-1} & 0 \\ 0 & 1 \end{matrix})\phi(t)d^*tda \\
&= \int_{K^*/F^*}\int_{F^*} \pi(\begin{matrix} a^{-1} & 0 \\ 0 & 1 \end{matrix})\phi(t)d^*tda \\
&= \int_{K^*/F^*}\int_{F^*} \phi(ta^{-1})d^*tda \\
&= \int_{K^*/F^*}\int_{F^*} \phi(ta)d^*tda \\
&= \int_{K^*} \phi(x)d^*x
\end{aligned} .$$

This implies that either $\gamma(\pi,\psi)$ is equal to one, or $\int_{K^*}\phi(x)d^*x$ is equal to zero on $D(K^*) \cap \pi(w)D(K^*)$. In the second case, we could define two independant K^*-invariant linear forms on $K(\pi,\psi) = D(K^*) + \pi(w)D(K^*)$, given by $\phi_1 + \pi(w)\phi_2 \mapsto \int_{K^*}\phi_1(x)d^*x$, and $\phi_1 + \pi(w)\phi_2 \mapsto \int_{K^*}\phi_2(x)d^*x$. This would contradict Theorem 1.2 of [A-G]. $\square$

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